APPENDIX. TELEPORTATION-BASED C-NOT GATE

Here, we describe in detail the scheme of a teleportation-based C-NOT gate. We give a step by step analyses of its implementation with our setup, shown in Fig. 2 in the main article.

We align each β-barium borate (BBO) crystal carefully to produce a pair of polarization entangled photons i and j in the state:

$$|\Psi^+\rangle_{ij} = \frac{1}{\sqrt{2}}\left(|H\rangle_i|H\rangle_j + |V\rangle_i|V\rangle_j\right) \tag{1}$$

We use the method described in ref. [14] to prepare the cluster state $|\chi\rangle$. Initially, photons 3, 4, 5 and 6 are in the state:

$$\begin{aligned}|\Psi^+\rangle_{34} &\otimes |\Psi^+\rangle_{56} \\ = &\frac{1}{2}\big(|H\rangle_3|H\rangle_4|H\rangle_5|H\rangle_6 + |H\rangle_3|H\rangle_4|V\rangle_5|V\rangle_6 + \\ &|V\rangle_3|V\rangle_4|H\rangle_5|H\rangle_6 + |V\rangle_3|V\rangle_4|V\rangle_5|V\rangle_6\big).\end{aligned} \tag{2}$$

We direct photons 4 and 6 to the two input modes of a polarization dependent beam splitter (PDBS), respectively. The transmission T_H (T_V) of horizontally (vertically) polarized light at the PDBS is 1 (1/3), and we thus get

$$\begin{aligned}\rightarrow\ &\frac{1}{2}\big(|H\rangle_3|H\rangle_{4'}|H\rangle_5|H\rangle_{6'} + \frac{1}{\sqrt{3}}|H\rangle_3|H\rangle_{4'}|V\rangle_5|V\rangle_{6'} \\ &+\frac{1}{\sqrt{3}}|V\rangle_3|V\rangle_{4'}|H\rangle_5|H\rangle_{6'} \\ &-\frac{1}{3}|V\rangle_3|V\rangle_{4'}|V\rangle_5|V\rangle_{6'}\big).\end{aligned} \tag{3}$$

Here we have neglected terms with more than one photon in a single output mode of the PDBS, since in the experiment we post select only terms that lead to a six-fold coincidence.

In order to symmetrize the state we place a PDBS' ($T_H = 1/3, T_V = 1$) in each output mode of the PDBS and receive

$$\begin{aligned}\rightarrow\ &\frac{1}{6}\big(|H\rangle_3|H\rangle_{4''}|H\rangle_5|H\rangle_{6''} + |H\rangle_3|H\rangle_{4''}|V\rangle_5|V\rangle_{6''} \\ &+|V\rangle_3|V\rangle_{4''}|H\rangle_5|H\rangle_{6''} - |V\rangle_3|V\rangle_{4''}|V\rangle_5|V\rangle_{6''}\big).\end{aligned} \tag{4}$$

This is already the desired four-qubit cluster state up to local unitary operations. To bring it to the desired form, we place half-wave plates (HWPs) – with an angle of 22.5° between the fast and the horizontal axis – into arms 3 and 4. This yields

$$\begin{aligned}\rightarrow\quad &\left(|H\rangle_3|H\rangle_{4''} + |V\rangle_3|V\rangle_{4''}\right)|H\rangle_5|H\rangle_{6''} \\ &+\left(|H\rangle_3|V\rangle_{4''} + |V\rangle_3|H\rangle_{4''}\right)|V\rangle_5|V\rangle_{6''} \\ = &\ |\chi\rangle_{34''56''},\end{aligned} \tag{5}$$

where we have neglected the overall pre-factor 1/6 and we arrive at the desired ancillary four-photon cluster state $|\chi\rangle$ described in ref. [10].

Photons 1 and 2 constitute the input to our C-NOT gate. We assume that they are in a most general input state $|\Psi_{in}\rangle_{12}$, where:

$$\begin{aligned}|\Psi_{in}\rangle_{ij} &= \alpha|H\rangle_i|H\rangle_j + \beta|H\rangle_i|V\rangle_j \\ &\quad+\gamma|V\rangle_i|H\rangle_j + \delta|V\rangle_i|V\rangle_j\end{aligned} \tag{6}$$

The pre-factors α, β, γ and δ are four arbitrary complex numbers satisfying $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Before we proceed, let us define the desired output state after a C-NOT operation:

$$\begin{aligned}|\Psi_{out}\rangle_{ij} &= U^{C-NOT}|\Psi_{in}\rangle_{ij} \\ &= \alpha|H\rangle_i|H\rangle_j + \beta|V\rangle_i|V\rangle_j + \\ &\quad\gamma|V\rangle_i|H\rangle_j + \delta|H\rangle_i|V\rangle_j\end{aligned} \tag{7}$$

The target qubit i is flipped on the condition that the control qubit j is in the state $|V\rangle$.

We can now express the combined state of all six photons in terms of Bell states for photons 1&3 and 2&5 and in terms of the desired output state $|\Psi_{out}\rangle_{46}$ for photons 4&6 with corresponding Pauli operations:

$$\begin{aligned}
&|\Psi_{in}\rangle_{12}\otimes|\chi\rangle_{3456} = \\
&\quad |\Phi^+\rangle_{13}|\Phi^+\rangle_{25}\ |\Psi_{out}\rangle_{46} + |\Phi^+\rangle_{13}|\Phi^-\rangle_{25}\ \hat{\sigma}_z^6|\Psi_{out}\rangle_{46} \\
&\quad +|\Phi^+\rangle_{13}|\Psi^+\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_x^6|\Psi_{out}\rangle_{46} + |\Phi^+\rangle_{13}|\Psi^-\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_x^6\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} \\
&\quad +|\Phi^-\rangle_{13}|\Phi^+\rangle_{25}\ \hat{\sigma}_z^4\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} + |\Phi^-\rangle_{13}|\Phi^-\rangle_{25}\ \hat{\sigma}_z^4|\Psi_{out}\rangle_{46} \\
&\quad +|\Phi^-\rangle_{13}|\Psi^+\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_z^4\hat{\sigma}_x^6\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} + |\Phi^-\rangle_{13}|\Psi^-\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_z^4\hat{\sigma}_x^6|\Psi_{out}\rangle_{46} \\
&\quad +|\Psi^+\rangle_{13}|\Phi^+\rangle_{25}\ \hat{\sigma}_x^4|\Psi_{out}\rangle_{46} + |\Psi^+\rangle_{13}|\Phi^-\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} \\
&\quad +|\Psi^+\rangle_{13}|\Psi^+\rangle_{25}\ \hat{\sigma}_x^6|\Psi_{out}\rangle_{46} + |\Psi^+\rangle_{13}|\Psi^-\rangle_{25}\ \hat{\sigma}_x^6\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} \\
&\quad +|\Psi^-\rangle_{13}|\Phi^+\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_z^4\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} + |\Psi^-\rangle_{13}|\Phi^-\rangle_{25}\ \hat{\sigma}_x^4\hat{\sigma}_z^4|\Psi_{out}\rangle_{46} \\
&\quad +|\Psi^-\rangle_{13}|\Psi^+\rangle_{25}\ \hat{\sigma}_z^4\hat{\sigma}_x^6\hat{\sigma}_z^6|\Psi_{out}\rangle_{46} + |\Psi^-\rangle_{13}|\Psi^-\rangle_{25}\ \hat{\sigma}_z^4\hat{\sigma}_x^6|\Psi_{out}\rangle_{46}
\end{aligned} \tag{8}$$

With the help of polarizing beam splitters, in our experiment we are able to identify the Bell states $|\Phi^{\pm}\rangle_{13}$ and $|\Phi^{\pm}\rangle_{25}$, i.e. we project the combined state of photons 1, 2, 3 and 5 onto one of the four possibilities $|\Phi^{\pm}\rangle_{13}|\Phi^{\pm}\rangle_{25}$. We thus have to consider four different results of the BSMs:

Result of BSMs	Output state
$\|\Phi^{+}\rangle_{13}\|\Phi^{+}\rangle_{25}$	$\|\Psi_{out}\rangle_{46}$
$\|\Phi^{+}\rangle_{13}\|\Phi^{-}\rangle_{25}$	$\hat{\sigma}_z^6\|\Psi_{out}\rangle_{46}$
$\|\Phi^{-}\rangle_{13}\|\Phi^{+}\rangle_{25}$	$\hat{\sigma}_z^4\hat{\sigma}_z^6\|\Psi_{out}\rangle_{46}$
$\|\Phi^{-}\rangle_{13}\|\Phi^{-}\rangle_{25}$	$\hat{\sigma}_z^4\|\Psi_{out}\rangle_{46}$

To receive the desired final state of photons 4 and 6, we have to apply corresponding Pauli operations, depending on the outcome of the BSMs.